A Numerical Study of Vector Absorbing Boundary Conditions for the Finite-Element Solution of Maxwell's Equations

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Abstract—**In three-dimensional vector solutions to Maxwell's equations, boundary conditions of the Bayliss–Turkel kind can be used to absorb outgoing radiation. The boundary conditions were used with curvilinear brick finite elements to analyze spherical test problems for which the exact fields are known. Errors due to incomplete absorbtion decrease as the outer boundary is moved further away. The second-order boundary condition is appreciably more accurate than the first-order, at the same cost.**

I. Introduction

THE finite element method is an effective way of solving Maxwell's equations numerically, particularly when dielectric and inhomogeneous materials are present. It is, for example, well-suited to the prediction of field intensities in living tissue, desirable in microwave hyperthermia treatment [1]. However, the method solves the equations in a finite volume. On the outer boundary of the volume, conditions must be imposed which represent the infinite free space beyond, into which electromagnetic waves may be flowing. One way of doing this is to couple to an integral equation on the boundary; this leads to a hybrid method that unfortunately lowers the sparsity of the finite-element matrices. An alternative is to use an absorbing boundary condition—a differential equation imposed on the boundary, and solved along with the equations within. Such absorbing conditions are only exactly correct when the boundary is at infinity. Nevertheless, they can be accurate enough at reasonable distances, and have the advantage of preserving sparsity.

The simplest absorbing boundary condition amounts to no more than the Sommerfeld radiation condition. However, higher order conditions exist that offer greater accuracy. The higher order forms were first developed for the scalar case [2]. Recently, three second-order vector conditions were presented [3]–[5]. Only one of these [4] is symmetric, i.e., preserves the symmetry of the finite-element matrix. To date, there has been very little numerical verification of any of these second-order conditions. In this letter, we give numerical results obtained using the symmetric form [4], including an examination of the reduction of error with the distance to the absorbing boundary.

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II. Theory

The solution to the problem is a stationary point of this functional:

$$F(H) = \int_V \left[\frac{1}{\epsilon_r}(\nabla \times H)^2 - k^2\mu_r H^2\right] dV + \int_{S_a} \left[jkH_t^2 + \beta(r)(\nabla \times H)_r^2 - \beta(r)(\nabla \cdot H_t)^2\right] dS. \quad (1)$$

H is the unknown, complex, vector, magnetic field. V is the volume of interest and S_a is a spherical surface separating V from the infinite volume of free-space outside. k is the free-space wavenumber. Subscripts r and t denote the r and $\theta - \phi$ parts of a vector, respectively, where (r, θ, ϕ) are spherical coordinates based on an origin at the centre of $S_a \beta(r)$ is $1/(2jk + 2/r)\epsilon_r$ and μ_r are the relative permittivity and permeability, respectively. Each may be complex, to represent loss, and inhomogeneous.

The second integral causes a second-order absorbing boundary condition to hold on S_a [4]. If $\beta(r)$ is replaced by zero, the first-order boundary condition applies instead.

In order to find the stationary points of F, the volume V is divided into curvilinear bricks known as covariant-projection elements [6]. These bricks represent H as a mixed first-/second-order polynomial, whose tangential part is continuous from one brick to the next. They have been shown to provide good solutions, free from the effects of spurious modes, and make the imposition of vector boundary conditions as straightforward as it is in the scalar case. The unknowns are the covariant components of H at points throughout V, and requiring that F be stationary results in a linear matrix equation for the unknowns: $Ax = b$, where A is a large, square, sparse, symmetric matrix. Solving for x allows H to be determined at any point in V.

Covariant-projection elements do not impose the normal continuity of H, because it is not generally required: the normal continuity of the magnetic flux density arises naturally from the variational principle. However, because of the divergence term $\nabla \cdot H_t$ in the surface integral, it *is* neces-

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TABLE I
BOUNDARY CONDITIONS FOR THE CASE $(m, n) = (1, 1)$
(r, θ, ϕ): $0.03\lambda \times 22.5° \times 22.5°$

Boundary Surface	Boundary Condition
$r = 0.3\lambda$	Prescribed H_θ and H_ϕ of the (1, 1) function
$r = R$	Absorbing boundary condition
$\theta = 3°$	Prescribed H_r and H_ϕ of the (1,1) function
$\theta = 90°$	Electric wall
$\phi = 0°$	Electric wall
$\phi = 90°$	Magnetic wall

TABLE II
BOUNDARY CONDITIONS FOR THE CASE $(m, n) = (1, 2)$
(r, θ, ϕ): $0.03\lambda \times 15° \times 18°$

Boundary Surface	Boundary Condition
$r = 0.3\lambda$	Prescribed H_θ and H_ϕ of the (1, 2) function
$r = R$	Absorbing boundary condition
$\theta = 45°$	Electric wall
$\theta = 90°$	Magnetic wall
$\phi = 0°$	Electric wall
$\phi = 90°$	Magnetic wall

sary to impose normal continuity on H on the surface only. The second-order results shown next were obtained in this way.

III. RESULTS

Harrington [7] gives expressions for spherical TE wave functions, which are exact solutions of Maxwell's equations in spherical coordinates. Each function is characterized by two integers, (m, n); the ϕ variation increases with m and the θ variation increases with n. For each (m, n), there are two functions, one representing an outgoing wave, and one an incoming wave.

Now consider a boundary-value problem in which H_θ and H_ϕ on the surface of a sphere are constrained to be as they are for a spherical wave function (m, n), and outside the sphere is infinite free space. The solution is a single, outgoing, spherical wave function (m, n).

This boundary-value problem was solved using the finite element method, and the results compared with the exact answers. The cases $(m, n) = (1, 1)$ and $(m, n) = (1, 2)$ are considered here. Both cases are three-dimensional in the sense that the field varies with r, θ and ϕ. Because of symmetry, only sectors of the sphere were modeled; details are given in Tables I and II. The curvilinear nature of the elements made them easy to fit to the spherical inner and outer boundaries (Fig. 1). Note that in the case (1, 1), to exclude the difficult line $\theta = 0$, it was necessary to make the field satisfy the exact solution on $\theta = 3°$ as well as on $r = 0.3\lambda$.

Figs. 2 and 3 show how the error changes as the outer boundary $r = R$ is moved outwards, for first and second-order absorbing boundary conditions. The error shown is the largest value of

$$e = |H_{\text{FEM}} - H_{\text{exact}}|$$

over the volume modeled, expressed as a percentage of

$$|H_{\text{exact}}|$$

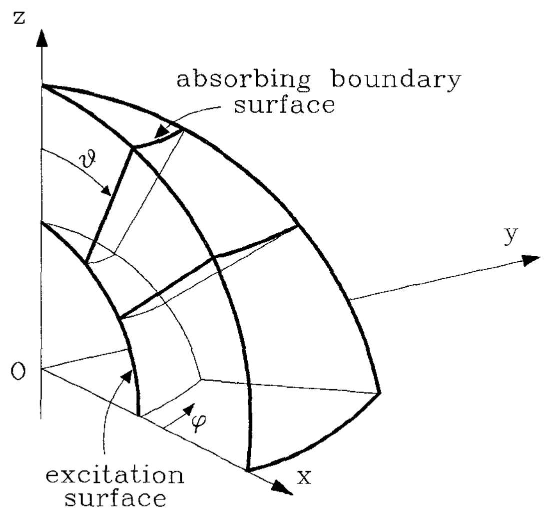

Fig. 1. Modeling a sector of a sphere with curvilinear bricks. Note the degenerate brick touching the z axis.

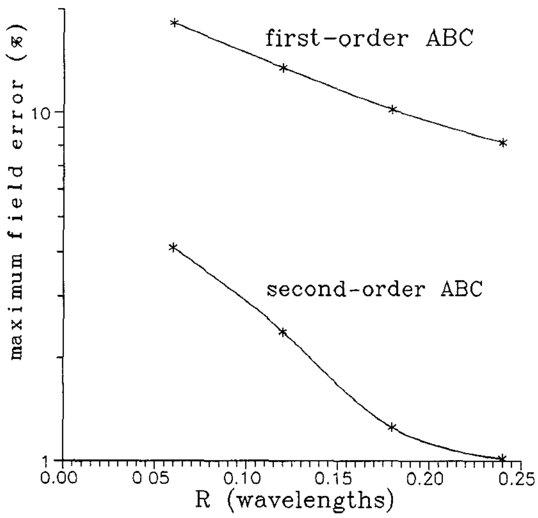

Fig. 2. Solution error versus R, the radius of the absorbing boundary, for the case $(m, n) = (1, 1)$.

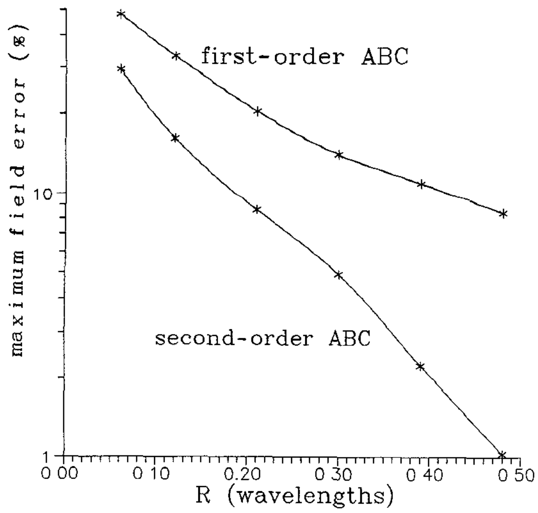

Fig. 3. Solution error versus R, the radius of the absorbing boundary, for the case $(m, n) = (1, 2)$.

at the point where the largest value of e occurs. Inaccuracies due to the discretization are estimated to be less than 1%, so the errors plotted arise almost entirely from the effects of the absorbing boundary.

Conclusion

The graphs show that, compared with the first-order absorbing boundary condition, the second-order condition gives errors that are smaller, and decrease faster as the outer boundary is moved away.

Since the computational cost of the two boundary conditions is almost the same, there can be no doubt that the second-order form is to be preferred. It allows reasonable accuracies to be obtained when the outer surface is still quite near.

Absorbing boundary conditions of order greater than two were not implemented. Such higher order conditions would introduce higher order derivatives, altering the sparsity pattern of the final matrix. Symmetric bilinear forms for orders greater than two have not been developed.

References

[1] K. D. Paulsen, D. R. Lynch, and J. W. Strohbehn, "Three dimensional finite, boundary and hybrid element solutions of the Maxwell equations for lossy dielectric media," *IEEE Trans. Microwave Theory Tech.*, vol. 36, pp. 682-693, Apr. 1988.

[2] A. Bayliss, M. Gunzburger, and E. Turkel, "Boundary conditions for the numerical solution of elliptic equations in exterior regions," *SIAM J. Appl. Math.*, vol. 42, no. 2, pp. 430-451, Apr. 1982.

[3] A. F. Peterson, "Absorbing boundary conditions for the vector wave equation," *Microwave and Optical Technol. Lett.*, vol. 1, no. 2, pp. 62-64, Apr. 1988.

[4] J. P. Webb and V. N. Kanellopoulos, "Absorbing boundary conditions for the finite element solution of the vector wave equation," *Microwave and Optical Technol. Lett.*, vol. 2, no. 10, pp. 370-372, Oct. 1989.

[5] J. D'Angelo and I. Mayergoyz, "Finite element methods for the solution of RF radiation and scattering problems," *Electromagn.*, vol. 10, pp. 177-199, 1990.

[6] C. W. Crowley, P. P. Silvester, and H. Hurwitz, "Covariant projection elements for 3-D vector field problems," *IEEE Trans. Magn.*, vol. 24, pp. 397-400, Jan. 1988.

[7] R. F. Harrington, *Time-Harmonic Electromagnetic Fields.* New York: McGraw-Hill, 1961, p. 269.